Assignment : 3

1. Find the Fourier transform of $f(x) = \begin{cases} \cos x, & |x| \le \pi \\ 0, & |x| > \pi. \end{cases}$

Solution :

We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos x\, e^{isx}\, dx \quad \left[\because f(x) = \begin{cases} \cos x, & -\pi < x < \pi \\ 0, & 0 \text{ therwise} \end{cases} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos x \left[\cos sx + i \sin sx\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos sx \cos x\, dx + \frac{i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos x \sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\pi} \cos sx \cos x\, dx + 0$$

since, $\cos sx \cos x$ is an even function

$\sin sx \sin x$ is an odd function.

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\pi} \left[\frac{\cos(s+1)x + \cos(s-1)x}{2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1}\right]_{0}^{\pi}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)\pi}{s+1} + \frac{\sin(s-1)\pi}{s-1} - (0+0)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s\pi + \pi)}{s+1} + \frac{\sin(s\pi - \pi)}{s-1}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(\pi + s\pi)}{s+1} - \frac{\sin(\pi - s\pi)}{s-1}\right]$$

Assignment-3

2) Find the Fourier transform of $f(x) = |x| e^{-|x|}$, $x \in R$.

<u>Solution</u>

$$f(x) = |x| e^{-|x|}$$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-|x|} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{-\infty}^{0} x e^{x} e^{isx} dx + \int_{0}^{\infty} x e^{-x} e^{isx} dx \right.$$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \left[-\int_{-\infty}^{0} x e^{(1+is)x} dx + \int_{0}^{\infty} x e^{-(1-is)x} dx \right]$$

$u = x$, $dv = e^{(1+is)x}$; $u' = 1$, $v_1 = \frac{e^{(1+is)x}}{(1+is)}$; $u'' = 0$, $v_2 = \frac{e^{(1+is)x}}{(1+is)^2}$

$u = x$, $v = e^{-(1-is)x}$; $u' = 1$, $v_1 = \frac{e^{-(1-is)x}}{-(1-is)}$; $u'' = 0$, $v_2 = \frac{e^{-(1-is)x}}{(1-is)^2}$

$$= \frac{1}{\sqrt{2\pi}} \left[-\left(x \frac{e^{(1+is)x}}{(1+is)} - \frac{e^{(1+is)x}}{(1+is)^2} \right)_{-\infty}^{0} + \left(-x \frac{e^{-(1-is)x}}{(1-is)} - \frac{e^{-(1-is)x}}{(1-is)^2} \right)_{0}^{\infty} \right]$$

$$= \frac{-1}{\sqrt{2\pi}} \left[\left(0 - \frac{1}{(1+is)^2}\right) - (0+0) + (0+0) - \left(0 + \frac{1}{(1-is)^2}\right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{(1+is)^2} + \frac{1}{(1-is)^2} \right) = \frac{1}{\sqrt{2\pi}} \left[\frac{(1-is)^2 + (1+is)^2}{(1+is)^2 (1-is)^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2 - 2s^2}{1 + 2s^2 + s^4} \right] = \sqrt{\frac{2}{\pi}} \left(\frac{1-s^2}{1+2s^2+s^4} \right)$$

Assignment-3

Q.No:3 Use Fourier Integral formula for

$$f(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases}$$ to find the value of the

Integral $\int_0^\infty \frac{\sin x \cos xt}{x}\, dx$.

Sol: We know that, $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1)\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} \sin sx\, dx.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 \cos sx\, dx$$ [$\because$ $\sin sx$ – odd fn, $\cos sx$ – even fn].

$$F[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^1 = \sqrt{\frac{2}{\pi}} \left(\frac{\sin s}{s}\right).$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin s}{s}\right) (\cos sx - i \sin sx)\, ds.$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} \cos sx\, ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} \sin sx\, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin s}{s} \cos sx\, ds$$ [$\because$ $\frac{\sin s}{s} \cos sx$ – even fn. $\frac{\sin s}{s} \sin sx$ – odd fn]

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin s \cos sx}{s}\, ds.$$

$$\Rightarrow \int_0^\infty \frac{\sin s \cos sx}{s}\, ds = \frac{\pi}{2} f(x).$$

Replace $s = x$, $x = t$, we get

$$\int_0^\infty \frac{\sin x \cos xt}{x}\, dx = \frac{\pi}{2} f(t)$$

$$= \begin{cases} \frac{\pi}{2}, & |t| < 1 \\ 0, & |t| > 1. \end{cases}$$

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Assignment 3

4) By Fourier Cosine integral,

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t \, dt \, d\lambda$$

$$f(t) = \begin{cases} \frac{\pi}{2}\sin t &, 0 < t < \pi \\ 0 &, t > \pi \end{cases}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\pi \frac{\pi}{2}\sin t \cos\lambda t \, dt \, d\lambda$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{2}\int_0^\infty \cos\lambda x \int_0^\pi \sin t \cos\lambda t \, dt \, d\lambda$$

$$= \int_0^\infty \cos\lambda x \int_0^\pi \sin t \cos\lambda t \, dt \, d\lambda$$

$$\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$= \int_0^\infty \cos\lambda x \int_0^\pi \left[\frac{\sin(1+\lambda)t + \sin(1-\lambda)t}{2}\right] dt \, d\lambda$$

$$= \frac{1}{2}\int_0^\infty \cos\lambda x \left\{\int_0^\pi \sin(1+\lambda)t + \sin(1-\lambda)t \, dt\right\} d\lambda$$

$$= \frac{1}{2}\int_0^\infty \cos\lambda x \left[-\frac{\cos(1+\lambda)t}{1+\lambda} - \frac{\cos(1-\lambda)t}{1-\lambda}\right]_0^\pi d\lambda$$

$$= \frac{1}{2}\int_0^\infty \cos\lambda x \left\{\left[-\frac{\cos(1+\lambda)\pi}{1+\lambda} - \frac{\cos(1-\lambda)\pi}{1-\lambda}\right] - \left[-\frac{1}{1+\lambda} - \frac{1}{1-\lambda}\right]\right\} d\lambda$$

$$= \frac{1}{2}\int_0^\infty \cos \lambda x \left[\left\{-\left(\frac{\cos\pi \cos\lambda\pi - \sin\pi \sin\lambda\pi}{1+\lambda}\right) \cancel{+} \left(\frac{\cos\pi \cos\lambda\pi + \sin\pi \sin\lambda\pi}{1-\lambda}\right)\right\} - \left\{-\frac{1}{1+\lambda} + \frac{1}{1-\lambda}\right\}\right] d\lambda$$

$$= \frac{1}{2}\int_0^\infty \cos\lambda x \left[-\frac{(-1)\cos\lambda\pi}{1+\lambda} + 0 + \frac{(-1)\cos\lambda\pi}{1-\lambda} + 0\right] - \left\{\frac{-(1-\lambda) + (1+\lambda)}{1-\lambda^2}\right\} d\lambda$$

$$= \frac{1}{2}\int_0^\infty \cos\lambda x \left[\frac{\cos\lambda\pi}{1+\lambda} + \frac{\cos\lambda\pi}{1-\lambda}\right] - \left[\frac{-1+\lambda - 1 - \lambda}{1-\lambda^2}\right] d\lambda$$

$$= \frac{1}{2}\int_0^\infty \cos\lambda x \; \frac{(1-\lambda)\cos\lambda\pi + (1+\lambda)\cos\lambda\pi}{1-\lambda^2} - \left(\frac{-2}{1-\lambda^2}\right)$$

$$= \frac{1}{2}\int_0^\infty \cos\lambda x \left(\frac{\cos\cancel{\lambda\pi} - \cancel{\lambda\cos\lambda\pi} + \cos\cancel{\lambda\pi} + \cancel{\lambda\cos\lambda\pi}}{1-\lambda^2}\right) + \frac{2}{1-\lambda^2}$$

$$= \frac{1}{2}\int_0^\infty \cos\lambda x \left[\frac{2\cos\lambda\pi + 2}{1-\lambda^2}\right] d\lambda$$

$$= \int_0^\infty \frac{\cos\lambda x}{1-\lambda^2}(1+\cos\pi\lambda)\, d\lambda.$$

Assignment - 3

5) Find the inverse fourier transform of the function

$$f(\xi) = \begin{cases} 2-|\xi|, & |\xi| \le 2 \\ 0, & |\xi| > 2 \end{cases}$$ and evaluate the integral $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$.

Solution Fourier Transform

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, e^{is\xi}\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} (2-|\xi|)(\cos s\xi + i \sin s\xi)\, d\xi$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^2 (2-\xi) \cos s\xi \, d\xi$$

$$= \frac{2}{\sqrt{2\pi}} \left[(2-\xi)\left(\frac{\sin s\xi}{s}\right) - (-1)\left(-\frac{\cos s\xi}{s^2}\right) \right]_0^2$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(0\frac{\sin 2s}{s} - \frac{\cos 2s}{s^2} \right) - \left(0 - \frac{1}{s^2} \right) \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left(\frac{1}{s^2} - \frac{\cos 2s}{s^2} \right) = \frac{2}{\sqrt{2\pi}} \left(\frac{1-\cos 2s}{s^2} \right)$$

$$= \frac{2}{\sqrt{2\pi}} \left(\frac{2\sin^2 s}{s^2} \right)$$

$$\boxed{F(s) = \frac{4}{\sqrt{2\pi}} \frac{\sin^2 s}{s^2}}$$

Inverse Fourier Transform,

$$f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\left(e^{-is\xi}\right) ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{4}{\sqrt{2\pi}} \sin^2 s \right) (\cos s\xi - i \sin s\xi)\, ds$$

$$= \frac{4}{2\pi} \times 2 \int_0^\infty \frac{\sin^2 s}{s^2} \cos s\xi \, ds$$

Put $\frac{t}{4}=0$ in above equation, we get

$$2-0=\frac{4}{\pi}\int_0^{\infty}\frac{\sin^2 s}{s^2}\,ds$$

$$2\times\frac{\pi}{4}=\int_0^{\infty}\frac{\sin^2 s}{s^2}\,ds$$

ie $$\int_0^{\infty}\frac{\sin^2 s}{s^2}\,ds=\frac{\pi}{2}$$

$$\therefore \boxed{\int_0^{\infty}\frac{\sin^2 x}{x^2}\,dx=\frac{\pi}{2}}$$ ($\because$ s is a dummy variable)